Spaces of strongly lacunary invariant summable sequances

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Abstract

In this paper, we introduce and examine some properties of three sequence spaces defined using lacunary sequence and invariant mean which generalize several known sequence spaces.

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1 Introduction

Let w be the set of all sequences real or complex and ℓ_∞ denote the Banach space of bounded sequences $\xi = \{\xi_k\}_{k=0}^{\infty}$ normed by $||\xi|| = \sup_{k\geq 0} |\xi_k|$. Lorentz [4] proved that

$$\hat{c} = \left\{ x : \lim_{m\to\infty} \frac{1}{m+1} \sum_{i=0}^{m} \xi_{n+i} \text{ exists uniformly in } n \right\}.$$

Let σ be a one-to-one mapping of the set of positive integers into itself. A continuous linear functional φ on l_∞ is said to be an invariant mean or a σ- mean if and only if

1. $\varphi(\xi) \geq 0$ when the sequence $\xi = (\xi_n)$ has $\xi_n \geq 0$ for all n.
2. $\varphi(e) = 1$, where $e = (1, 1, \ldots)$ and
3. $\varphi\left(\xi_{\sigma(n)}\right) = \varphi(\xi)$ for all $\xi \in l_\infty$.

For a certain kinds of mapping σ every invariant mean φ extends the limit functional on space c, in the sense that $\varphi(\xi) = \lim \xi$ for all $\xi \in c$. Consequently, $c \subset V_\sigma$ where V_σ is the bounded sequences all of whose σ-means are equal, (see, [17]).

If $\xi = (\xi_k)$, set $T\xi = (T\xi_k) = \left(\xi_{\sigma(k)}\right)$ it can be shown that (see, Schaefer [12]) that

$$V_\sigma = \left\{ \xi \in l_\infty \ : \ \lim_k t_{km}(\xi) = Le \text{ uniformly in m for some } L = \sigma - \lim \xi \right\} \tag{1.1}$$

where

$$t_{km}(\xi) = \frac{\xi_m + T\xi_m + \ldots + T^k \xi_m}{k+1} \quad \text{and} \quad t_{-1,m} = 0$$

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We say that a bounded sequence $\xi = (\xi_k)$ is σ-convergent if and only if $\xi \in V_\sigma$ such that $\sigma^k(n) \neq n$ for all $n \geq 0$, $k \geq 1$.

Just as the concept of almost convergence lead naturally to the concept of strong almost convergence, σ- convergence leads naturally to the concept of strong σ-convergence. A sequence $\xi = (\xi_k)$ is said to be strongly σ-convergent (see Mursaleen [8]) if there exists a number L such that

$$\frac{1}{k}\sum_{i=1}^{k}\left|\xi_{\sigma^i(m)} - L\right| \to 0 \tag{1.2}$$

as $k \to \infty$ uniformly in m. We write $[V_\sigma]$ as the set of all strong σ- convergent sequences. When (1.2) holds we write $[V_\sigma] - \lim x = L$. Taking $\sigma(m) = m+1$, we obtain $[V_\sigma] = [\hat{c}]$, which is defined in [5], so strong σ- convergence generalizes the concept of strong almost convergence. Note that

$$[V_\sigma] \subset V_\sigma \subset l_\infty.$$

Spaces of strongly summable sequences were discussed by Kuttner [3], Maddox [5] and others. The invariant summable sequences have been discussed by Schafer [17] and some others. Mursaleen [9] have considered absolute invariant convergent and absolute invariant summable sequences. Also the strongly invariant summable sequences was studied by Saraswat and Gupta[11]. Some works related to invariant summable sequences can be found in [10, 12, 13, 14, 15]. The goal of this paper is to study the spaces of strongly lacunary $\sigma-$ summable sequences, which naturally come up for investigation and which will fill up a gap in the existing literature.

Let $\theta = (k_r)$ be the sequence of positive integers such that

i) $k_0 = 0$ and $0 < k_r < k_{r+1}$

ii) $h_r = (k_r - k_{r-1}) \to \infty$ as $r \to \infty$.

Then θ is called a lacunary sequence. The intervals determined by θ are denoted by $I = (k_r - k_{r-1}]$. The ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r (see, Freedman et al [2]).

Recently, Das and Mishra [1] defined M_θ, the set of almost lacunary convergent sequences, as follows:

$$M_\theta = \left\{\xi : \text{ there exists } l \text{ such that uniformly in } i \geq 0, \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}(\xi_{k+i} - l) = 0\right\}.$$

Let $T = (t_{nk})$ be an infinite matrix of nonnegative real numbers and $p = (p_k)$ be a sequence such that $p_k > 0$. We write $T\xi = \{T_n(\xi)\}$ if $T_n(\xi) = \sum_k t_{nk}\left|\xi_k\right|^{p_k}$ converges for each n. We write.

$$d_{rn}(\xi) = \frac{1}{h_r}\sum_{i\in I_r} T_{\sigma^n(i)}(\xi) = \sum_k t(n,k,r)\left|\xi_k\right|^{p_k}$$

where

$$t(n,k,r) = \frac{1}{h_r}\sum_{i\in I_r} t_{\sigma^n(i),k}.$$

If we take $\sigma(n) = n + 1$

$$d_{rn}(\xi) = \frac{1}{h_r} \sum_{i \in I_r} T_{\sigma^n(i)}(\xi) = \sum_k t(n,k,r) \left|\xi_k\right|^{p_k}$$

and

$$t(n,k,r) = \frac{1}{h_r} \sum_{i \in I_r} t_{\sigma^n(i)}, k.$$

reduces to

$$t_{rn}(\xi) = \frac{1}{h_r} \sum_{i=0}^{m} T_{n+i}(\xi) = \sum_k t(n,k,r) \left|\xi_k\right|^{p_k}$$

where

$$t(n,k,r) = \frac{1}{r} \sum_{i=0}^{m} t_{n+i,k}.$$

We define the following sequence spaces:

$$\begin{aligned} \left[T_{(\theta,\sigma)}, p\right]_0 &= \{\xi : d_{rn}(\xi) \to 0 \text{ uniformly in } n\}; \\ \left[T_{(\theta,\sigma)}, p\right] &= \{\xi : d_{rn}(\xi - le) \to 0 \text{ for some } l \text{ uniformly in } n\} \end{aligned}$$

and

$$\left[T_{(\theta,\sigma)}, p\right]_\infty = \left\{x : \sup_n d_{rn}(\xi) < \infty\right\}.$$

The sets $[T_{(\theta,\sigma)}, p]_0$, $[T_{(\theta,\sigma)}, p]$ and $[T_{(\theta,\sigma)}, p]_\infty$ will be respectively called the spaces of strongly lacunary σ -summable to zero, strongly lacunary σ -summable and strongly lacunary σ- bounded sequences. If $\sigma(n) = n + 1$, the above spaces reduces to the following sequence spaces which are introduced in [16].

$$\begin{aligned} \left[\hat{T}_\theta, p\right]_0 &= \{\xi : t_{rn}(\xi) \to 0 \text{ uniformly in } n\}; \\ \left[\hat{T}_\theta, p\right] &= \{\xi : t_{rn}(\xi - le) \to 0 \text{ for some } l \text{ uniformly in } n\} \end{aligned}$$

and

$$\left[\hat{T}_\theta, p\right]_\infty = \left\{\xi : \sup_n t_{rn}(\xi) < \infty\right\}.$$

If ξ is strongly lacunary σ- summable to l we write $\xi_k \to l[T_{(\theta,\sigma)}, p]$. A pair (T, p) will be called strongly lacunary σ - invariant regular if

$$\xi_k \to l \Rightarrow \xi_k \to l[T_{(\theta,\sigma)}, p].$$

2 The main results

Before giving main theorem we have some propositions:

Proposition 2.1. If $p \in \ell_\infty$, then $[T_{(\theta,\sigma)}, p]_0$, $[T_{(\theta,\sigma)}, p]$ and $[T_{(\theta,\sigma)}, p]_\infty$ are linear spaces over $\mathbb{C}$.

Proof. We consider only $[T_{(\theta,\sigma)}, p]$. If $H = \sup p_k$ and $K = \max(1, 2^{H-1})$ we have [see Maddox [5], p. 346].

$$|t_k + b_k|^{p_k} \leqq K(|a_k|^{pk} + |b_k|^{pk}) \tag{2.1}$$

and for $\lambda \in \mathbb{C}$,

$$|\lambda|^{p_k} \leqq \max(1, |\lambda|^H). \tag{2.2}$$

Suppose that $\xi_k \to l[T_{(\theta,\sigma)}, p]$, $\rho_k \to l[T_{(\theta,\sigma)}, p]$ and $\lambda, \mu \in \mathbb{C}$. Then we have

$$d_{rn}(\lambda\xi + \mu\rho - (\lambda l + \mu \acute{l})e) \leqq KK_1 d_{rn}(\xi - le) + KK_2 d_{mn}(\rho - \acute{l}e)$$

where $K_1 = \sup |\lambda|^{p_k}$ and $K_2 = \sup |\mu|^{p_k}$, and this implies that $\lambda\xi + \mu\rho \to \left(\lambda l + \mu \acute{l}\right)[T_{(\theta,\sigma)}, p]$. This completes the proof. ∎

We have

Proposition 2.2. $[T_{(\theta,\sigma)}, p] \subset [T_{(\theta,\sigma)}, p]_\infty$, if

$$||T|| = \sup_r \sum_k t(n,k,r) < \infty. \tag{2.3}$$

Proof. Assume that $\xi_k \to l[T_{(\theta,\sigma)}, p]$ and (2.3) holds. Now by the inequality (2.1),

$$\begin{aligned} d_{rn}(\xi) &= d_{mn}(\xi - le + le) \\ &\leqq K d_{rn}(\xi - le) + K \sum_k t(n,k,r) |l|^{p_k} \\ &\leqq K d_{rn}(\xi - le) + K(\sup |l|^{p_k}) \sum_k t(n,k,r). \end{aligned} \tag{2.4}$$

Therefore $\xi \in [T_{(\theta,\sigma)}, p]_\infty$ and this completes the proof. ∎

Remark 2.3. Some known sequence spaces are obtained by specializing T and therefore some of the results proved here extend the corresponding results obtained for the special cases.

Proposition 2.4. Let $p \in \ell_\infty$, then $[T_{(\theta,\sigma)}, p]_0$ and $[T_{(\theta,\sigma)}, p]_\infty$ ($\inf p_k > 0$) are linear topological spaces paranormed by g defined by

$$g(\xi) = \sup_{r,n} [d_{rn}(\xi)]^{1/M}$$

where $M = \max(1, H = \sup p_k)$. If (2.3) holds, then $[T_{(\theta,\sigma)}, p]$ has the same paranorm.

Proof. Clearly $g(0)=0$ and $g(\xi)=g(-\xi)$. Since $M \geqq 1$, by Minkowski's inequality it follows that g is subadditive. We now show that the scalar multiplication is continuous. It follows from the inequality (2.2) that

$$g(\lambda\xi) \leqq \sup |\lambda|^{p_k/M}\, g(\xi).$$

Therefore $\xi \to 0 \Rightarrow \lambda\xi \to 0$ (for fixed λ). Now let $\lambda \to 0$ and ξ be fixed. Given $\varepsilon > 0$ $\exists N$ such that

$$d_{rn}(\lambda\xi) < \varepsilon/2\, (\forall n, \forall r > N)\,. \tag{2.5}$$

Since $d_{r,n}(\xi)$ exists for all r, we write

$$d_{rn}(\xi) = K(r), (1 \leqq r \leqq N)$$

and

$$\delta = \left(\frac{\varepsilon}{2K(r)}\right)^{1/p_k}.$$

Then $|\lambda| < \delta$,

$$d_{rn}(\lambda\xi) < \frac{\varepsilon}{2}\, (\forall n, 1 \leqq r \leqq N)\,. \tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\lambda \to 0 \Rightarrow \lambda\xi \to 0 \;\; (\xi \text{ fixed})$$

This proves the assertion about $[T_{(\theta,\sigma)}, p]_0$. If $\inf p_k = \theta > 0$ and $0 < |\lambda| < 1$, then $\forall \xi \in [T_{(\theta,\sigma)}, p]_\infty$,

$$g^M(\lambda\xi) \leqq |\lambda|^\theta\, g^M(\xi).$$

Therefore $[T\theta, p]_\infty$ has the paranorm g. If (2.3) holds it is clear from Proposition 2.2 that $g(\xi)$ exists for each $\xi \in [T_{(\theta,\sigma)}, p]$. This completes the proof. ∎

Remark 2.5. It is clear that g is not a norm in general. But if $p_k = p \;\forall k$,then clearly g is a norm for $1 \leqq p \leqq \infty$ and a $p-$ norm for $0 < p < 1$.

Proposition 2.6. $[T_{(\theta,\sigma)}, p]_0$ and $[T_{(\theta,\sigma)}, p]_\infty$ are complete with respect to their paranorm topologies $[T_{(\theta,\sigma)}, p]$ is complete if (2.3) holds and

$$\sum_k t(n,k,r) \to 0 \text{ uniformly in } n. \tag{2.7}$$

The proof is easy and we omit it.

Combining the above proposition we have the main result.

Theorem 2.7. Let $p \in \ell_\infty$. Then $[T_{(\theta,\sigma)}, p]_0$ and $[T_{(\theta,\sigma)}, p]_\infty$ $(\inf p_k > 0)$ are complete linear topological spaces paranormed by g. If (2.3) and (2.7) hold then $[T_{(\theta,\sigma)}, p]$ has the same property. If further $p_k = p$ for all k, they are Banach spaces for $1 \leqq p < \infty$ and $p-$normed spaces for $0 < p < 1$.

We now give locally boundedness and $q-$convexity for the spaces of strongly almost summable sequences. We start with some definitions. For $< q \leqq 1$ a non-void subset U of a linear space is said to be absolutely $q-$convex if $\xi, \rho \in U$ and $|\gamma|^q + |\mu|^q \leq 1$ together imply that $\gamma\xi + \mu\rho \in U$. It is clear that if U is absolutely $q-$convex, then it is absolutely $t-$ convex for $t < q$. A linear topological space X is said to be $q-$ convex if every neighbourhood of $0 \in X$ contains an absolutely $q-$convex neighbourhood of $0 \in X$. The $q-$convexity for $q > 1$ is of little interest, since X is $q-$convex for $q > 1$ if and only if X is the only neighbourhood of $0 \in X$, [see Maddox and Roles [6]]. A subset B of X is said to be bounded if for each neighbourhood U of $0 \in X$ there exists an integer $N > 1$ such that $B \subseteq NU$. X is called locally bounded if there is a bounded neighbourhood of zero.

We have

Theorem 2.8. Let $0 < p_k \leqq 1$. Then $[T_{(\theta,\sigma)}, p]_0$ and $[T_{(\theta,\sigma)}, p]_\infty$ are locally bounded if $\inf p_k > 0$. If (2.3) holds, then $[T_{(\theta,\sigma)}, p]$ has the same property.

The proof of the above theorem follows on the same lines as adopted by Savas [16]. So we omitted it.

It is known that every locally bounded linear topological space is $q-$ convex for some q such that $0 < q \leqq 1$. But the following theorem gives exact conditions for $q-$convexity.

Theorem 2.9. Let $0 < p_k \leqq 1$.Then $\left[T_{(\theta,\sigma)}, p\right]_0$ and $\left[T_{(\theta,\sigma)}, p\right]_\infty$ are $q-$convex for all q where $0 < q < \liminf p_k$. Moreover, if $p_k = p \leqq 1\ \forall k$, then they are $p-$convex. $\left[T_{(\theta,\sigma)}, p\right]$ has the same properties if (2.3) holds.

Proof. We shall prove the theorem only for $\left[T_{(\theta,\sigma)}, p\right]_\infty$. Let $\left[T_{(\theta,\sigma)}, p\right]_\infty$ and $q \in (0, \liminf p_k)$. Then $\exists k_0$ such that $q \leqq p_k$ $(\forall k > k_0)$. Now define

$$g\sigma(\xi) = \sup_{r,n} \left[\sum_{k=1}^{k_0} t(n,k,r)\, |\xi_k|^q + \sum_{k=k_0+1}^{\infty} t(n,k,r)\, |\xi_k|^{p_k} \right].$$

Since $q \leqq p_k \leqq 1$ $(\forall k > k_0)$, $g\sigma$ is subadditive. Further for $0 < |\gamma| \leqq 1$,

$$|\gamma|^{p_k} \leqq |\gamma|^q \quad (\forall k > k_0)$$

Therefore for such γ we have

$$g\sigma(\gamma\xi) \leqq |\gamma|^q\, g\sigma(x).$$

Now for $0 < \delta < 1$,

$$U = \{x : g\sigma(\xi) \leqq \delta\}$$

is an absolutely $q-$convex set, for $|\gamma|^q + |\mu|^q \leqq 1$ and $\xi, \rho \in U$ imply that

$$\begin{aligned} g\sigma(\gamma\xi + \mu\rho) &\leqq g\sigma(\gamma\xi) + g\sigma(\mu\rho) \leqq |\gamma|^q\, g\sigma(\xi) + |\mu|^q\, g\sigma(\rho) \\ &\leqq (|\gamma|^q + |\mu|^q)\, \delta \leqq \delta. \end{aligned}$$

If $p_k = p$ $(\forall k)$, then for $0 < \delta < 1$,

$$V = \{x : g\sigma \leqq \delta\}$$

is an absolutely $p-$convex set. This can be prove by the same analysis and therefore we omit the details. This completes the proof. ∎

References

[1] G. Das and S. K. Mishra, *Banach limits and lacunary strong almost convergence*, J. Orissa Math. Soc. 2(2), (1983), 61-70.

[2] A. R. Freedman, J. J. Sember and M. Rapheal, *Some Cesaro-type summability spacces*, Proc. London Math. Soc. (3) 37 (1973), 508-520.

[3] B. Kuttner (1946), *Note on strong summability*, J. London Math. Soc. 21, 118-22.

[4] G. G. Lorentz (1948), *A contribution to the theory of divergent sequences*, Acta Math. 80, 167-190.

[5] I. J. Maddox , *Spaces of strongly summable sequences*, Quart. J. Math. Oxford Ser. (2) 18, 345-55.

[6] I. J. Maddox and J. W. Roles, *Absolute convexity in certain topological linear spaces*, Proc. Camb. Philos. Soc. 66,(1969), 541-45.

[7] I. J. Maddox (1970), Elements of Functional Analysis (Camb. Univ. Press).

[8] Mursaleen, *Matrix transformation between some new sequence spaces*, Houston J. Math. **9**(1993), 505–509.

[9] Mursaleen, *On some new invariant matrix methods of summability*, Q.J. Math. 34 (1983), 77-86.

[10] F. Nuray and E.Savas, *Some new sequence spaces defined by a modulus function*, Indian J. Pure Appl. Math., 24 (4), (1993), 657-663.

[11] S. K. Saraswat and S. K. Gupta, *Spaces of strongly σ-summable sequences*, Bull. Cal. Math. Soc. 75,(1983), 179-184,

[12] E. Savaş, *A note on absolute σ -summability*, Istanbul Univ. Fac. Sci. Math. J. 50 ,(1991), 123-128.

[13] E. Savaş,*Invariant means and generalization of a theorem of S. Mishra*, Doa Trk. J. Math. 14, (1989), 8-14.

[14] E. Savaş, *On strong σ-convergence*, J. Orissa Math. Soc. Vol. 5, No.2, (1986), 45-53.

[15] E. Savaş, *On lacunary strong σ-convergence*, Indian J. Pure Appl. Math., 21 (4), (1990), 359-365.

[16] E. Savaş, *Lacunary almost convergence and some new sequence spaces*, Filomat 33 (5), (2019), 13971402.

[17] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc. **36**(1972), 104–110.